Online Appendix for "Multiple Testing and the Distributional Effects of Accountability Incentives in Education"

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Abstract

This is the online appendix for Lehrer, Pohl, and Song (2021). Three sections are included, which provide further details regarding (1) additional motivation for the testing procedure, (2) the asymptotic validity of the multiple testing procedure, and (3) robustness checks for the main result.

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A Additional motivation for the testing procedure from an empirical perspective

The well-documented diverse and heterogeneous behavior in how individuals respond to a particular treatment or intervention has not only changed how economists think about econometric models and policy evaluation but also has profound consequences for the scientific evaluation of public policy. James Heckman stresses this point in his 2001 Nobel lecture, where he notes that conditional mean impacts including the average treatment effect may provide limited guidance for policy design and implementation (Heckman, 2001). Although the importance of heterogeneous treatment effects is widely recognized in the causal inference literature, common practice remains to report an average causal effect parameter. While an increasing number of studies account for possible treatment effect heterogeneity when evaluating programs or other interventions, most conduct statistical inference without allowing for dependence across subgroups.

As Fink, McConnell, and Vollmer (2014) point out, a majority of studies based on field experiments published in 10 specific journals estimate separate average causal parameters for different subgroups, but report traditional standard errors and p-values when testing for heterogeneous treatment effects through interaction terms or subgroup analyses. This is inappropriate because each interaction term represents a separate hypothesis beyond the original experimental design and results in a substantially increased type I error. The problem when testing multiple hypotheses jointly is the potential over-rejection of the null hypothesis. Intuitively, if the null hypothesis of no treatment effect is true, testing it across 100 subsamples, we expect about five rejections at the 95 percent confidence level. However, since the probability of a false positive equals 0.05 for each individual hypothesis, the probability of falsely rejecting at least one true null hypothesis may be much larger. Hence, the type I error exceeds the nominal size of the test.

A similar observation can be made for distributional treatment effects. A growing number of studies examine if treatment effects differ across quantiles of the outcome variable, i.e. they estimate quantile treatment effects (QTEs) (e.g., Heckman, Smith, and Clements, 1997; Friedlander and Robins, 1997; Abadie, 2002; Bitler, Gelbach, and Hoynes, 2006; Firpo, 2007). Testing for the presence of positive (or, generally, non-zero) QTEs involves a test of multiple hypotheses, for example 99 hypotheses in the case of percentile treatment effects. Therefore, the naive approach of comparing individual test results to find quantile groups with positive and statistically significant treatment effects inevitably suffers from the issue of data mining due to the reuse of the same data as emphasized by White (2000). As a result, the type I error rates can exceed the desired level of the test, which leads researchers to reject "too many" individual hypotheses.[1] To the best of our knowledge, no published study estimating distributional treatment effects makes such a correction. Among articles published in five high-impact economic journals between 2008 to 2017 that estimate distributional treatment effects none corrects inference for multiple testing (Allen, Clark, and Houde, 2014; Angrist, Lang, and Oreopoulos, 2009; Bandiera et al., 2017; Banerjee et al., 2015; Behaghel, de Chaisemartin, and Gurgand, 2017; Brown et al., 2014; Crepon et al., 2015; Evans and Garthwaite, 2012; Fack and Landais, 2010; Fairlie and Robinson, 2013; McKenzie, 2017; Meyer and Sullivan, 2008; Muralidharan, Niehaus, and Sukhtankar, 2016). The absence of these corrections may reflect that econometric testing procedures for QTEs were not previously developed. Lehrer, Pohl, and Song (2021) aims to fill that gap and also provide a formal result of asymptotic validity when the propensity scores are parametrically specified.

B Mathematical proofs

Throughout the proof, we assume that $B = \infty$ for simplicity, so that we ignore the bootstrap simulation errors in the proof. Our first result is the asymptotic linear representation of $\sqrt{n}(\hat{q}_d(\tau) - q_d(\tau))$ and its bootstrap version that is uniform over $\tau \in [\tau_L, \tau_U]$. Let us introduce some notation. Let

$$a_\tau(Y_i; q) = \tau - 1\{Y_i \leqslant q\}, \tag{1}$$

and for $u \in \mathbb{R}$,

$$\Delta(Y_i; q, u) = \sqrt{n} \int_0^1 \left(a_\tau(Y_i; q + n^{-1/2} us) - a_\tau(Y_i; q) \right) ds. \tag{2}$$

Note that the right hand side in (2) does not depend on τ.

Let $J_d(\tau_U, \tau_L) = \{q_d(\tau) : \tau \in [\tau_L, \tau_U]\}$, and assume that it is bounded in $\mathbb{R}$. (See Assumption 2.3(ii) of the main text.) For $q \in J_d(\tau_U, \tau_L)$ and $u \in \mathbb{R}$, let

$$\varphi_n(Y_i; q, u) = \frac{\Delta(Y_i; q, u)}{n^{3/4}} = -n^{-1/4} \int_0^1 1\{q < Y_i \leqslant q + n^{-1/2} us\} ds. \tag{3}$$

[1] In part as a response, statistical inference procedures developed in Heckman, Smith, and Clements (1997), Abadie, Angrist, and Imbens (2002), Rothe (2010), and Maier (2011), among others, focus on the whole distribution of potential outcomes to side-step multiple comparisons.

Let $\mathcal{B}$ be a class of bounded measurable functions $b : \mathbb{R} \times \mathcal{X} \times \{0,1\} \to \mathbb{R}$. Define for each $u \in \mathbb{R}$,

$$\mathcal{H}_n(u) = \{\varphi_n(\cdot; q, u)b(\cdot) : (q,b) \in J_d(\tau_U, \tau_L) \times \mathcal{B}\}. \tag{4}$$

We let $V_i = (Y_i, X_i', D_i) \in \mathbb{R}^{d_V}$ for brevity of notation.

We introduce a pseudo-norm $\|\cdot\|_{P,2}$ on the set of measurable functions on $\mathbb{R} \times \mathcal{X} \times \{0,1\}$: $\|f\|_{P,2} = (E|f(V_i)|^2)^{1/2}$, for any measurable map f. We also denote the sup norm by $\|\cdot\|_\infty$: $\|f\|_\infty = \sup_{v \in \mathbb{R}^{d_V}} |f(v)|$. For each $\varepsilon > 0$ and $u \in \mathbb{R}$, let $N_{[]}(\varepsilon, \mathcal{H}_n(u), \|\cdot\|_{P,2})$ denote the ε-bracketing number of $\mathcal{H}_n(u)$ with respect to $\|\cdot\|_{P,2}$ (see van der Vaart and Wellner, 1996, p. 83).

Lemma B.1 *For $u \in \mathbb{R}$, there exist constants $C_1, C_2, C_3, C_4 > 0$ such that for each $\varepsilon \in (0,1)$, there are brackets $[h_{L,j}, h_{U,j}]$ with $1 \leqslant j \leqslant N(\varepsilon)$, satisfying that the brackets cover $\mathcal{H}_n(u)$ and for each $k \geqslant 2$,*

$$E[|h_{L,j}(V_i) - h_{U,j}(V_i)|^k] \leqslant C_1 (C_2 n^{-1/4})^{k-2} \varepsilon^2, \tag{5}$$

and

$$\log N(\varepsilon) \leqslant C_3 - C_3 \log(\varepsilon) + C_4 \log N_{[]}(C\varepsilon^2, \mathcal{B}, \|\cdot\|_2). \tag{6}$$

Proof: First, define for $\delta > 0$,

$$\begin{aligned} z_\delta(y; q, s, u) &= (1 - \min\{(y - q - n^{-1/2}us)/\delta, 1\})1\{0 < y - q - n^{-1/2}us\} \\ &\quad + 1\{y - q - n^{-1/2}us \leqslant 0\}. \end{aligned} \tag{7}$$

Define

$$\varphi_{U,\delta}(y; q, u) = \int_0^1 \varphi_{U,\delta}(y; q, s, u) ds, \text{ and} \tag{8}$$

$$\varphi_{L,\delta}(y; q, u) = \int_0^1 \varphi_{L,\delta}(y; q, s, u) ds, \tag{9}$$

where

$$\varphi_{U,\delta}(y; q, s, u) = z_\delta(y; q, s) - z_\delta(y + \delta + n^{-1/2}us; q, s), \text{ and} \tag{10}$$

$$\varphi_{L,\delta}(y; q, s, u) = \min\left\{z_\delta(y + \delta; q, s), z_\delta(-y + 2q + \delta + n^{-1/2}us; q, s)\right\}. \tag{11}$$

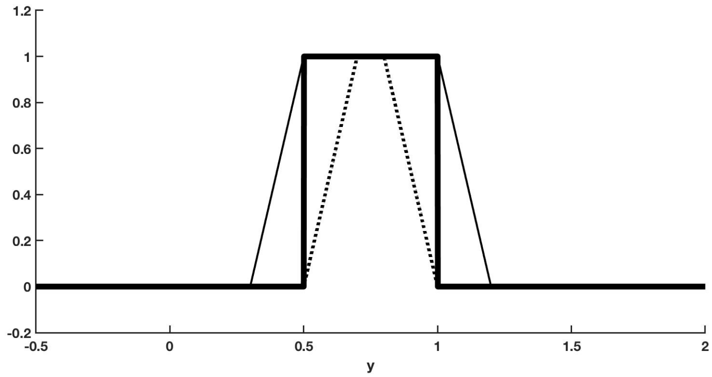


Figure 1: Illustration of $\varphi_{U,\delta}, \varphi_{L,\delta}$: The solid thick line depicts $\varphi(\cdot; q, s, u)$, the solid thin line $\varphi_{U,\delta}(\cdot; q, s, u)$ and the dotted line $\varphi_{L,\delta}(\cdot; q, s, u)$. Here we take $n^{-1/2}us = 0.5, q = 0.5$ and $\delta = 0.2$. The absolute slope of both maps $\varphi_{U,\delta}(\cdot; q, s, u)$ and $\varphi_{L,\delta}(\cdot; q, s, u)$ are bounded by $1/\delta$.

Let $\varphi(y; q, s, u) = 1\{y - q - n^{-1/2}us \leqslant 0\} - 1\{y - q \leqslant 0\}$ and define

$$\varphi_n(y; q, u) = n^{-1/4} \int_0^1 \varphi(y; q, s, u) ds. \tag{12}$$

Note that the definition (3) conforms with this.

Then, we have for all $y \in \mathbb{R}$, (see Figure 1)

$$\varphi_{L,\delta}(y; q, u) \leqslant n^{1/4} \varphi_n(y; q, u) \leqslant \varphi_{U,\delta}(y; q, u). \tag{13}$$

It is not hard to see that for all $q, q' \in \mathbb{R}$, and all $y \in \mathbb{R}$,

$$\begin{aligned} |\varphi_{U,\delta}(y; q, u) - \varphi_{U,\delta}(y; q', u)| &\leqslant |q - q'|/\delta, \text{ and} \\ |\varphi_{L,\delta}(y; q, u) - \varphi_{L,\delta}(y; q', u)| &\leqslant |q - q'|/\delta. \end{aligned} \tag{14}$$

Furthermore, for some constant $C > 0$,

$$E\left[(\varphi_{U,\delta}(Y_i; q, u) - \varphi_{L,\delta}(Y_i; q, u))^2 \,|D_i = d \right] \leqslant C\delta, \tag{15}$$

and

$$E\left[\varphi_{U,\delta}^2(Y_i; q, u) | D_i = d \right] \leqslant 1, \text{ and } E\left[\varphi_{L,\delta}^2(Y_i; q, u) | D_i = d \right] \leqslant 1. \tag{16}$$

Define

$$\mathcal{H}_{L,\delta}(u) = \{\varphi_{L,\delta}(\cdot;q,u)b(\cdot)/n^{1/4} : (q,b) \in J_d(\tau_U,\tau_L) \times \mathcal{B}\}, \text{ and} \tag{17}$$
$$\mathcal{H}_{U,\delta}(u) = \{\varphi_{U,\delta}(\cdot;q,u)b(\cdot)/n^{1/4} : (q,b) \in J_d(\tau_U,\tau_L) \times \mathcal{B}\}. \tag{18}$$

From (14) and using the fact that $n \geqslant 1$, $J_d(\tau_U,\tau_L)$ is bounded, and $\varphi_{L,\delta}(\cdot;q,u)$, $\varphi_{U,\delta}(\cdot;q,u)$ and $b(\cdot)$ are bounded maps, we find that

$$\begin{aligned} N_{[]}(\varepsilon, \mathcal{H}_{L,\delta}(u), \|\cdot\|_{P,2}) &\leqslant C(\varepsilon\delta)^{-1} \times N_{[]}(C\varepsilon, \mathcal{B}, \|\cdot\|_{P,2}), \text{ and} \\ N_{[]}(\varepsilon, \mathcal{H}_{U,\delta}(u), \|\cdot\|_{P,2}) &\leqslant C(\varepsilon\delta)^{-1} \times N_{[]}(C\varepsilon, \mathcal{B}, \|\cdot\|_{P,2}), \end{aligned} \tag{19}$$

for all $\varepsilon > 0$, for some constant $C > 0$. We take $\delta = \varepsilon^2$ and ε^2-brackets $[h_{L,a,j}, h_{L,b,j}]_{j=1}^N$ and $[h_{U,a,j}, h_{U,b,j}]_{j=1}^N$ such that the former set of brackets cover $\mathcal{H}_{L,\varepsilon^2}(u)$ and the latter $\mathcal{H}_{U,\varepsilon^2}(u)$, both with respect to $\|\cdot\|_{P,2}$, and

$$\|h_{U,a,j}\|_\infty + \|h_{U,b,j}\|_\infty + \|h_{L,a,j}\|_\infty + \|h_{L,b,j}\|_\infty \leqslant \frac{C}{n^{1/4}}, \tag{20}$$

for all $j = 1, ..., N$, for some constant $C > 0$. By (13) and (19), we lose no generality by taking brackets so that for each $h \in \mathcal{H}_n(u)$, there exists $j \in \{1, ..., N\}$ such that[2]

$$\min\{h_{U,b,j}, h_{L,a,j}\} \leqslant h \leqslant \max\{h_{L,a,j}, h_{U,b,j}\}, \tag{21}$$

and

$$\log N \leqslant C - C\log\varepsilon + C\log N_{[]}(C\varepsilon^2, \mathcal{B}, \|\cdot\|_{P,2}), \tag{22}$$

for some $C > 0$. We set

$$h_{L,j} = \min\{h_{U,b,j}, h_{L,a,j}\}, \text{ and } h_{U,j} = \max\{h_{U,b,j}, h_{L,a,j}\}. \tag{23}$$

[2] Since b can take negative values, the inequality (13) does not necessarily imply that $h_{L,a,j} \leqslant h \leqslant h_{U,b,j}$.

Therfore, by (20) and (15), for each $k \geqslant 2$,

$$\begin{aligned} E[|h_{L,j}(V_i) - h_{U,j}(V_i)|^k] &\leqslant C(Cn^{-1/4})^{k-2} E\left[(h_{L,a,j}(V_i) - h_{U,b,j}(V_i))^2\right] \\ &\leqslant 2C(Cn^{-1/4})^{k-2} E\left[(h_{L,a,j}(V_i) - h_{L,b,j}(V_i))^2\right] \\ &\quad +4C(Cn^{-1/4})^{k-2} E\left[(h_{L,b,j}(V_i) - h_{U,a,j}(V_i))^2\right] \\ &\quad +4C(Cn^{-1/4})^{k-2} E\left[(h_{U,a,j}(V_i) - h_{U,b,j}(V_i))^2\right] \\ &\leqslant C_1(C_2 n^{-1/4})^{k-2}\left(\varepsilon^4 + \varepsilon^2 + \varepsilon^4\right), \end{aligned} \tag{24}$$

for some constants $C, C_1, C_2 > 0$. The terms ε^4 are due to the choice of ε^2-brackets and the term ε^2 comes from (15) and $\delta = \varepsilon^2$. ■

Define for each $\tau \in [\tau_L, \tau_U]$ and $b \in \mathcal{B}$,

$$U(\tau, b; \delta) = \{(\tau_1, b_1) \in [\tau_L, \tau_U] \times \mathcal{B} : |\tau - \tau_1| + \|b - b_1\|_{P,2} \leqslant \delta\}. \tag{25}$$

Recall the definitions of $a_\tau(Y_i; q)$ and $\Delta(Y_i; q, u)$ in (1) and (2).

Lemma B.2 *Suppose that $\mathcal{B}$ and $\mathcal{H}_n(u)$ are as in Lemma B.1, for some $u \in \mathbb{R}$, and that for $d = 0, 1$, the density f_d of Y_{di} is bounded. Furthermore, assume that there exists $C > 0$ such that for all $\varepsilon > 0$,*

$$\log N_{[]}(\varepsilon, \mathcal{B}, \|\cdot\|_{P,2}) \leqslant C - C\log\varepsilon. \tag{26}$$

Then the following statements hold.

(i) There exist $s > 0$ and $C > 0$ such that for all $n \geqslant 1$, and for all $\delta > 0$,

$$E\left[\sup_{(\tau_1,b_1)\in U(\tau,b;\delta)}\left|\frac{1}{\sqrt{n}}\sum_{i=1}^n \left(a_{\tau_1}(Y_i; q_d(\tau_1))b_1(V_i) - E\left[a_{\tau_1}(Y_i; q_d(\tau_1))b_1(V_i)\right]\right)\right|\right] \leqslant C\delta^s.$$

(ii) There exists $C > 0$ such that for all $n \geqslant 1$,

$$E\left[\sup_{h\in\mathcal{H}_n(u)}\left|\sum_{i=1}^n \left(h(V_i) - E\left[h(V_i)\right]\right)\right|\right] \leqslant C\log n.$$

Proof: (i) Let

$$\mathcal{B}_1 = \{a_\tau(\cdot; q) : (\tau, q) \in [\tau_L, \tau_U] \times \mathbb{R}\}. \tag{27}$$

Certainly, $\mathcal{B}_1$ is a VC class. Both classes $\mathcal{B}_1$ and $\mathcal{B}$ are classes of bounded functions. Hence, if we let $\mathcal{B}_2$ be the collection of functions $f(\cdot)g(\cdot)$ as we run $f \in \mathcal{B}_1$ and $g \in \mathcal{B}$, we have for some constant $C > 0$,

$$\log N_{[]}(\varepsilon, \mathcal{B}_2, \|\cdot\|_{P,2}) \leqslant C - C \log \varepsilon + \log N_{[]}(C\varepsilon, \mathcal{B}, \|\cdot\|_{P,2}). \tag{28}$$

Using (26), we obtain the finite integral bracketing entropy bound for the left hand side of (28). The desired result of (i) follows by the maximal inequality. (For example, see (1) in van der Vaart (1996).)

(ii) When $u = 0$, we have $\Delta(Y_i; q_d(\tau), u) = 0$, a.s. We focus on the case $u \neq 0$. Observe that since b is bounded and $|\varphi_n(Y_i; q, u)| \leqslant 1$, for some constant $C > 0$,

$$E\left[|\varphi_n(Y_i; q, u)b(V_i)|^k\right] \leqslant CE\left[|\varphi_n(Y_i; q, u)|^k\right] \leqslant CE\left[\varphi_n^2(Y_i; q, u)\right], \tag{29}$$

for all $k \geqslant 2$. Observe that

$$\begin{aligned}
E\left[\varphi_n^2(Y_i; q, u)\right] &\leqslant \frac{1}{\sqrt{n}} \int_0^1 P\{q \leqslant Y_i \leqslant q + n^{-1/2} us\} ds & (30)\\
&= \frac{1}{\sqrt{n}} \int_0^1 P\{q \leqslant Y_{1i} \leqslant q + n^{-1/2} us | D_i = 1\} P\{D_i = 1\} ds \\
&\quad + \frac{1}{\sqrt{n}} \int_0^1 P\{q \leqslant Y_{0i} \leqslant q + n^{-1/2} us | D_i = 0\} P\{D_i = 0\} ds \\
&\leqslant \frac{1}{\sqrt{n}} \int_0^1 P\{q \leqslant Y_{1i} \leqslant q + n^{-1/2} us\} ds & (31)\\
&\quad + \frac{1}{\sqrt{n}} \int_0^1 P\{q \leqslant Y_{0i} \leqslant q + n^{-1/2} us\} ds \\
&\leqslant \frac{2}{n} \max_{d=0,1} \sup_{q \in \mathbb{R}} f_d(q) u.
\end{aligned}$$

Using this in combination with Lemma B.1, we apply Theorem 6.8 of Massart (2007) (taking $b = 1$ and $\sigma = Cn^{-1/2}$ there) to obtain that

$$\begin{aligned}
E\left[\sup_{h \in \mathcal{H}_n(u)} \left|\sum_{i=1}^n (h(V_i) - Eh(V_i))\right|\right] &\leqslant C_1 + C_1\sqrt{n} \int_0^{C_1/\sqrt{n}} \sqrt{\log(1/z)} dz + C_1 \log n \\
&\leqslant C_2 \log n,
\end{aligned}$$

for some constants $C_1, C_2 > 0$ from large n on. Thus we obtain the desired result. ■

Lemma B.3 *For $d = 0, 1$,*

$$\sum_{i=1}^{n} \frac{(\tau - 1\{Y_i \leqslant \hat{q}_d(\tau)\})1\{D_i = d\}}{\hat{p}_d(X_i)} = 0. \tag{32}$$

Proof: Note Knight's identity (see (15) on page 1822 of Kato (2009))

$$\rho_\tau(x - y) - \rho_\tau(x) = -y(\tau - 1\{x \leqslant 0\}) + y \int_0^1 (1\{x \leqslant ys\} - 1\{x \leqslant 0\})ds. \tag{33}$$

Take any $\varepsilon > 0$. By the definition of $\hat{q}_d(\tau)$ as a minimizer of $\hat{Q}_d(q, \tau)$ over q defined in Section 2.2.3,

$$0 \leqslant \hat{Q}_d(\hat{q}_d(\tau) - \varepsilon, \tau) - \hat{Q}_d(\hat{q}_d(\tau), \tau) \tag{34}$$

$$= -\varepsilon \sum_{i=1}^{n} \frac{(\tau - 1\{Y_i \leqslant \hat{q}_d(\tau)\})1\{D_i = d\}}{\hat{p}_d(X_i)} \tag{35}$$

$$+ \varepsilon \sum_{i=1}^{n} \frac{\int_0^1 (1\{Y_i \leqslant \hat{q}_d(\tau) + \varepsilon s\} - 1\{Y_i \leqslant \hat{q}_d(\tau)\})ds 1\{D_i = d\}}{\hat{p}_d(X_i)} \tag{36}$$

and

$$0 \leqslant \hat{Q}_d(\hat{q}_d(\tau) + \varepsilon, \tau) - \hat{Q}_d(\hat{q}_d(\tau), \tau) \tag{37}$$

$$= \varepsilon \sum_{i=1}^{n} \frac{(\tau - 1\{Y_i \leqslant \hat{q}_d(\tau)\})1\{D_i = d\}}{\hat{p}_d(X_i)} \tag{38}$$

$$- \varepsilon \sum_{i=1}^{n} \frac{\int_0^1 (1\{Y_i \leqslant \hat{q}_d(\tau) - \varepsilon s\} - 1\{Y_i \leqslant \hat{q}_d(\tau)\})ds 1\{D_i = d\}}{\hat{p}_d(X_i)}. \tag{39}$$

Hence

$$\sum_{i=1}^{n} \frac{\int_0^1 (1\{Y_i \leqslant \hat{q}_d(\tau) - \varepsilon s\} - 1\{Y_i \leqslant \hat{q}_d(\tau)\})ds 1\{D_i = d\}}{\hat{p}_d(X_i)} \tag{40}$$

$$\leqslant \sum_{i=1}^{n} \frac{(\tau - 1\{Y_i \leqslant \hat{q}_d(\tau)\})1\{D_i = d\}}{\hat{p}_d(X_i)} \tag{41}$$

$$\leqslant \sum_{i=1}^{n} \frac{\int_0^1 (1\{Y_i \leqslant \hat{q}_d(\tau) + \varepsilon s\} - 1\{Y_i \leqslant \hat{q}_d(\tau)\})ds 1\{D_i = d\}}{\hat{p}_d(X_i)}. \tag{42}$$

By sending $\varepsilon \to 0$, we obtain the desired result. ■

Theorem B.1 *Suppose that Assumptions 2.2 and 2.3 in the main text hold, and let*

$$\zeta_i = \psi(V_i) - E\psi(V_i), \text{ and } \zeta_i^* = \psi(V_i^*) - E[\psi(V_i^*)|\mathcal{F}_n]. \tag{43}$$

Then the following statements hold.

(i)

$$\begin{aligned}
&\sqrt{n}(\hat{q}_d(\tau) - q_d(\tau)) \\
&= \frac{1}{\sqrt{n} f_d(q_d(\tau))} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \\
&+ \frac{1}{\sqrt{n} f_d(q_d(\tau))} \sum_{j=1}^{n} E\left[\frac{a_\tau(Y_i; q_d(\tau)) g_d(X_i; \beta_0)' 1\{D_i = d\}}{p_d^2(X_i)}\right] \zeta_j + o_P(1),
\end{aligned}$$

uniformly over $\tau \in [\tau_L, \tau_U]$.

(ii)

$$\begin{aligned}
&\sqrt{n}(\hat{q}_d^*(\tau) - \hat{q}_d(\tau)) \\
&= \frac{1}{\sqrt{n} f_d(q_d(\tau))} \sum_{j=1}^{n} \left(\frac{a_\tau(Y_j^*; q_d(\tau)) 1\{D_j = d\}}{p_d(X_j^*)} - \frac{1}{n} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \right) \\
&+ \frac{1}{\sqrt{n} f_d(q_d(\tau))} \sum_{j=1}^{n} E\left[\frac{a_\tau(Y_i; q_d(\tau)) g_d(X_i; \beta_0)' 1\{D_i = d\}}{p_d^2(X_i)}\right] \zeta_j^* + o_P(1),
\end{aligned}$$

uniformly over $\tau \in [\tau_L, \tau_U]$.

Proof: (i) Note that by the definition of $\hat{q}_d(\tau)$,

$$\begin{aligned}
\sqrt{n}(\hat{q}_d(\tau) - q_d(\tau)) &= \arg\min_{u \in \mathbb{R}} \hat{Q}_d(q_d(\tau) + n^{-1/2}u; \tau) \\
&= \arg\min_{u \in \mathbb{R}} \left(\hat{Q}_d(q_d(\tau) + n^{-1/2}u; \tau) - \hat{Q}_d(q_d(\tau); \tau) \right).
\end{aligned}$$

Recall the definitions of $\hat{Q}_d(q;\tau)$ and $Q_d(q;\tau)$ in Section 2.2.3 of the main text. We write

$$\hat{Q}_d(q_d(\tau) + n^{-1/2}u; \tau) - \hat{Q}_d(q_d(\tau); \tau) = A_n + B_n, \tag{44}$$

where

$$
\begin{aligned}
A_n &= \hat{Q}_d(q_d(\tau) + n^{-1/2}u;\tau) - \hat{Q}_d(q_d(\tau);\tau) &(45)\\
&\quad - \left(Q_d(q_d(\tau) + n^{-1/2}u;\tau) - Q_d(q_d(\tau);\tau)\right), \text{ and} &(46)\\
B_n &= Q_d(q_d(\tau) + n^{-1/2}u;\tau) - Q_d(q_d(\tau);\tau).
\end{aligned}
$$

We can follow the same arguments as in the proof of Theorem 3 of Kato (2009), and show that

$$
B_n = -\frac{u}{\sqrt{n}} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau))1\{D_i = d\}}{p_d(X_i)} + \frac{u^2}{2} f_d(q_d(\tau)) + o_P(1), \tag{47}
$$

uniformly over $\tau \in [\tau_L, \tau_U]$. Let us focus on A_n. Using Knight's identity in (33), we write A_n as

$$
uZ_{n,d}^{(1)}(\tau) + Z_{n,d}^{(2)}(u,\tau), \tag{48}
$$

where

$$
\begin{aligned}
Z_{n,d}^{(1)}(\tau) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau))1\{D_i = d\}}{p_d(X_i)} \left(\frac{p_d(X_i)}{\hat{p}_d(X_i)} - 1\right), \text{ and} &(49)\\
Z_{n,d}^{(2)}(u,\tau) &= -\frac{u}{n} \sum_{i=1}^{n} \left(\frac{p_d(X_i)}{\hat{p}_d(X_i)} - 1\right) \frac{\Delta(Y_i; q_d(\tau), u)1\{D_i = d\}}{p_d(X_i)}. &(50)
\end{aligned}
$$

We write

$$
Z_{n,d}^{(1)}(\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau))1\{D_i = d\}}{p_d(X_i)} \frac{p_d(X_i) - \hat{p}_d(X_i)}{p_d(X_i)} + R_n(\tau), \tag{51}
$$

where

$$
R_n(\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{a_\tau(Y_i; q_d(\tau))1\{D_i = d\}}{p_d(X_i)} \frac{(p_d(X_i) - \hat{p}_d(X_i))^2}{p_d(X_i)\hat{p}_d(X_i)}. \tag{52}
$$

By expanding $G(x;\hat{\beta})$ around β_0 and using Assumption 2.3 (i), it is not hard to see that

$$
\hat{p}_d(x) - p_d(x) = g_d(x;\beta_0)'(\hat{\beta} - \beta_0) + O_P(n^{-1}), \tag{53}
$$

uniformly over $x \in \mathcal{X}$. Since $|a_\tau(Y_i; q_d(\tau))/p_d(X_i)| \leqslant \varepsilon^{-1}$ for all $\tau \in [\tau_L, \tau_U]$ by Assumption 2.1(ii), we find that

$$\sup_{\tau \in [\tau_L, \tau_U]} |R_n(\tau)| = O_P(n^{-1/2}). \tag{54}$$

Applying this and the expansion in (53) to the leading term on the right hand side of (51), we obtain that

$$\begin{aligned} Z_{n,d}^{(1)}(\tau) &= -\frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'(\hat{\beta} - \beta_0)}{p_d(X_i)} + o_P(1) \\ &= -\left(\frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'}{p_d(X_i)} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i + o_P(1) \right) + o_P(1) \\ &= -E\left[\frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'}{p_d(X_i)} \right] \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i + o_P(1), \end{aligned}$$

by Lemma B.2(i). Using the same arguments, we also obtain that

$$Z_{n,d}^{(2)}(u, \tau) = -u \left(\frac{1}{n} \sum_{i=1}^n \frac{\Delta(Y_i; q_d(\tau), u) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'}{p_d(X_i)} \right) \frac{1}{n} \sum_{i=1}^n \zeta_i + o_P(1) = o_P(1).$$

We define $b_0(V_i) = 1\{D_i = d\} g_{d,k}(X_i; \beta_0)/p_d^2(X_i)$, where $g_{d,k}(X_i; \beta_0)$ is the k-th entry of $g_d(X_i; \beta_0)$, and take $\mathcal{B} = \{b_0\}$, i.e., the singleton of b_0 in the definition of $\mathcal{H}_n(u)$ in (4). We bound

$$\begin{aligned} \sup_{\tau \in [\tau_L, \tau_U]} & \Bigg| \frac{1}{n} \sum_{i=1}^n \frac{\Delta(Y_i; q_d(\tau), u) 1\{D_i = d\}}{p_d(X_i)} \frac{g_{d,k}(X_i; \beta_0)}{p_d(X_i)} \\ & \qquad - E\left[\frac{\Delta(Y_i; q_d(\tau), u) 1\{D_i = d\}}{p_d(X_i)} \frac{g_{d,k}(X_i; \beta_0)}{p_d(X_i)} \right] \Bigg| \\ & \leqslant n^{-1/4} \sup_{h \in \mathcal{H}_n(u)} \left| \sum_{i=1}^n (h(V_i) - Eh(V_i)) \right|. \end{aligned}$$

By Lemma B.2(ii), we find that uniformly over $\tau \in [\tau_L, \tau_U]$,

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \frac{\Delta(Y_i; q_d(\tau), u) 1\{D_i = d\}}{p_d(X_i)} \frac{g_{d,k}(X_i; \beta_0)}{p_d(X_i)} &= E\left[\frac{\Delta(Y_i; q_d(\tau), u) 1\{D_i = d\}}{p_d(X_i)} \frac{g_{d,k}(X_i; \beta_0)}{p_d(X_i)} \right] \\ &\quad + O(n^{-1/4} \log n). \end{aligned}$$

Since $E[\zeta_i] = 0$, we find that

$$Z_{n,d}^{(2)}(u,\tau) = o_P(1), \text{ uniformly over } \tau \in [\tau_L, \tau_U]. \tag{55}$$

Therefore, we conclude that

$$A_n = -E\left[\frac{a_\tau(Y_i; q_d(\tau))1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i;\beta_0)'}{p_d(X_i)}\right] \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i + o_P(1). \tag{56}$$

Combining this with (47), and applying Theorem 2 of Kato (2009), we obtain the desired result of (i).

(ii) The proof of the bootstrap version is similar to that of (i). First, we write

$$\sqrt{n}(\hat{q}_d^*(\tau) - \hat{q}_d(\tau)) = \arg\min_{u \in \mathbb{R}} \hat{Q}_d^*(\hat{q}_d(\tau) + n^{-1/2}u; \tau) - \hat{Q}_d^*(\hat{q}_d(\tau); \tau), \tag{57}$$

where

$$\hat{Q}_d^*(q;\tau) = \sum_{i=1}^n \frac{1\{D_i^* = d\}}{\hat{p}_d^*(X_i^*)} \rho_\tau(Y_i^* - q), \text{ and} \tag{58}$$

$$Q_d^*(q;\tau) = \sum_{i=1}^n \frac{1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} \rho_\tau(Y_i^* - q). \tag{59}$$

Similarly as before, we write

$$\hat{Q}_d^*(\hat{q}_d(\tau) + n^{-1/2}u; \tau) - Q_d^*(\hat{q}_d(\tau); \tau) = A_n^* + B_n^*, \tag{60}$$

where

$$\begin{aligned} A_n^* &\equiv \hat{Q}_d^*(\hat{q}_d(\tau) + n^{-1/2}u; \tau) - \hat{Q}_d^*(\hat{q}_d(\tau); \tau) && (61)\\ &\quad -(Q_d^*(\hat{q}_d(\tau) + n^{-1/2}u; \tau) - Q_d^*(\hat{q}_d(\tau)), \text{ and} && (62)\\ B_n^* &\equiv Q_d^*(\hat{q}_d(\tau) + n^{-1/2}u; \tau) - Q_d^*(\hat{q}_d(\tau); \tau). \end{aligned}$$

Following the similar arguments as before, we obtain that

$$A_n^* = -uE\left[\frac{a_\tau(Y_i^*; \hat{q}_d(\tau))1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} \frac{g_d(X_i^*;\hat{\beta})'}{\hat{p}_d(X_i^*)} |\mathcal{F}_n\right] \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i^* + o_P(1). \tag{63}$$

Note that from (i) of this theorem and Assumption 2.2(i),

$$\sup_{\tau \in [\tau_L, \tau_U]} |\hat{q}_d(\tau) - q_d(\tau)| = o_P(1), \text{ and } \hat{\beta} = \beta_0 + o_P(1). \tag{64}$$

Hence using Lemma B.2(i), we obtain that

$$\begin{aligned} & E\left[\frac{a_\tau(Y_i^*; \hat{q}_d(\tau)) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} \frac{g_d(X_i^*; \hat{\beta})'}{\hat{p}_d(X_i^*)} | \mathcal{F}_n \right] \\ = \; & \frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; \hat{q}_d(\tau)) 1\{D_i = d\}}{\hat{p}_d(X_i)} \frac{g_d(X_i; \hat{\beta})'}{\hat{p}_d(X_i)} \\ = \; & \frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'}{p_d(X_i)} + o_P(1) \\ = \; & E\left[\frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \frac{g_d(X_i; \beta_0)'}{p_d(X_i)} \right] + o_P(1). \end{aligned} \tag{65}$$

Let us turn to B_n^* defined in (61). Using Knight's identity, we write B_n^* as

$$u Z_{n,d}^{*(1)}(\tau) + Z_{n,d}^{*(2)}(u, \tau), \tag{66}$$

where

$$Z_{n,d}^{*(1)}(\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{a_\tau(Y_i^*; \hat{q}_d(\tau)) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)}, \text{ and} \tag{67}$$

$$Z_{n,d}^{*(2)}(u, \tau) = -\frac{u}{n} \sum_{i=1}^n \frac{\hat{\Delta}(Y_i^*; \hat{q}_d(\tau), u) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)}, \tag{68}$$

with

$$\hat{\Delta}(Y_i^*; \hat{q}_d(\tau), u) = \sqrt{n} \int_0^1 \left(a_\tau(Y_i^*; \hat{q}_d(\tau) + n^{-1/2} u s) - a_\tau(Y_i^*; \hat{q}_d(\tau)) \right) ds. \tag{69}$$

Let us first consider $Z^{*(2)}_{n,d}(u,\tau)$. Recall the notation $V_i = (Y_i, X_i', D_i)$ and $V_i^* = (Y_i^*, X_i^{*\prime}, D_i^*)$. We write $Z^{*(2)}_{n,d}(u,\tau)$ as

$$-\frac{u}{n}\sum_{i=1}^{n}\left(\tilde{\eta}(V_i^*;\hat{q}_d(\tau),\hat{\beta}) - \frac{1}{n}\sum_{i=1}^{n}\tilde{\eta}(V_i;\hat{q}_d(\tau),\hat{\beta})\right) \tag{70}$$

$$-\frac{u}{n}\sum_{i=1}^{n}\left(\tilde{\eta}(V_i;\hat{q}_d(\tau),\hat{\beta}) - \int\tilde{\eta}(v;\hat{q}_d(\tau),\hat{\beta})dF_V(v)\right)$$

$$-u\left(\int\tilde{\eta}(v;\hat{q}_d(\tau),\hat{\beta})dF_V(v) - \int\tilde{\eta}(v;q_d(\tau),\beta_0)dF_V(v)\right)$$

$$-uE\left[\frac{\Delta(Y_i;q_d(\tau))1\{D_i = d\}}{p_d(X_i)}\right], \tag{71}$$

where F_V is the CDF of V_i, and

$$\tilde{\eta}(V_i^*;\hat{q}_d(\tau),\hat{\beta}) = \frac{\Delta(Y_i^*;\hat{q}_d(\tau))1\{D_i^* = d\}}{\hat{p}_d(X_i^*)}. \tag{72}$$

We show that the first two terms in (70) are $o_P(1)$ uniformly over $\tau \in [\tau_L, \tau_U]$. We will deal with the first term in (70). By Assumptions 2.1(ii) and 2.3(i) in the main text, we can find $\varepsilon > 0$ such that $G_d(x;\beta) > 0$ for all $x \in \mathcal{X}$ and all $\beta \in B(\beta_0;\varepsilon)$, where $B(\beta_0;\varepsilon) = \{\beta \in \Theta : \|\beta - \beta_0\| \leqslant \varepsilon\}$. Define

$$b_\beta(V_i) = \frac{1\{D_i = d\}}{G_d(X_i;\beta)}, \tag{73}$$

and let

$$\mathcal{B} = \{b_\beta : \beta \in B(\beta_0;\varepsilon)\}, \tag{74}$$

and define $\mathcal{H}_n(u)$ as in (4) using this $\mathcal{B}$. Since the set $B(\beta_0;\varepsilon)$ is bounded in $\mathbb{R}^{d_\beta}$, by Assumptions 2.1(ii) and 2.3(i) in the main text, we find that the bracketing condition in (26) is satisfied for this set $\mathcal{B}$. Furthermore, by Assumption 2.2(i) in the main text, we have

$\hat{\beta} \in B(\beta_0; \varepsilon)$ with probability approaching one. Now, observe that

$$\begin{aligned}
&\sup_{\tau \in [\tau_L, \tau_U]} \left| \frac{1}{n} \sum_{i=1}^n \left(\frac{\Delta(Y_i^*; \hat{q}_d(\tau), u) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} \right. \right. \qquad (75)\\
&\qquad \left. \left. - E\left[\frac{\Delta(Y_i^*; \hat{q}_d(\tau), u) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} | \mathcal{F}_n \right] \right) \right| \\
&\leqslant n^{-1/4} \sup_{h \in \mathcal{H}_n(u)} \left| \sum_{i=1}^n (h(V_i^*) - E[h(V_i^*)|\mathcal{F}_n]) \right| = O_P(n^{-1/4} \log n),
\end{aligned}$$

by Lemma B.2(ii). Thus the first term in (70) is $o_P(1)$ uniformly over $\tau \in [\tau_L, \tau_U]$. The second term can be dealt with in the same way.

Let us turn to the third term in (70). This term is also $o_P(1)$ because $\hat{q}_d(\tau) = q_d(\tau) + o_P(1)$ and $\hat{\beta} = \beta_0 + o_P(1)$. Following precisely the same argument in the proof of Theorem 3 in Kato (2009) used to deal with B_n in (45), we can show that

$$-uE\left[\frac{\Delta(Y_i; q_d(\tau)) 1\{D_i = d\}}{p_d(X_i)} \right] = \frac{u^2}{2} f_d(q_d(\tau)) + o(1), \qquad (76)$$

uniformly over $\tau \in [\tau_L, \tau_U]$.

As for $Z_{n,d}^{*(1)}(\tau)$, we use Lemma B.3, and write

$$Z_{n,d}^{*(1)}(\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{a_\tau(Y_i^*; \hat{q}_d(\tau)) 1\{D_i^* = d\}}{G_d(X_i^*; \hat{\beta})} - \frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; \hat{q}_d(\tau)) 1\{D_i = d\}}{G_d(X_i; \hat{\beta})} \right).$$

Using Lemma B.2(i) and similar arguments used to show (75) above, we can show that

$$Z_{n,d}^{*(1)}(\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{a_\tau(Y_i^*; q_d(\tau)) 1\{D_i^* = d\}}{G_d(X_i^*; \beta_0)} - \frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{G_d(X_i; \beta_0)} \right) + o_P(1).$$

Hence collecting the results for $Z_{n,d}^{*(1)}(\tau)$ and $Z_{n,d}^{*(2)}(u, \tau)$, we conclude that

$$\begin{aligned}
&\hat{Q}_d^*(\hat{q}_d(\tau) + n^{-1/2} u; \tau) - Q_d^*(\hat{q}_d(\tau); \tau) \\
&= -uE\left[\frac{a_\tau(Y_i^*; \hat{q}_d(\tau)) 1\{D_i^* = d\}}{\hat{p}_d(X_i^*)} \frac{g_d(X_i^*; \hat{\beta})'}{\hat{p}_d(X_i^*)} | \mathcal{F}_n \right] \frac{1}{\sqrt{n}} \sum_{i=1}^n \zeta_i^* \\
&- u \frac{1}{\sqrt{n}} \sum_{i=1}^n \left(\frac{a_\tau(Y_i^*; q_d(\tau)) 1\{D_i^* = d\}}{G_d(X_i^*; \beta_0)} - \frac{1}{n} \sum_{i=1}^n \frac{a_\tau(Y_i; q_d(\tau)) 1\{D_i = d\}}{G_d(X_i; \beta_0)} \right) \\
&+ \frac{u^2}{2} f_d(q_d(\tau)) + o_P(1).
\end{aligned}$$

Now the desired result follows from Theorem 2 of Kato (2009) similarly as before. ■

Let us define

$$q^{\Delta}(\tau) = q_1(\tau) - q_0(\tau), \text{ and } \hat{q}^{\Delta}(\tau) = \hat{q}_1(\tau) - \hat{q}_0(\tau). \tag{77}$$

Similarly, we define a bootstrap version $\hat{q}^{\Delta *}(\tau) = \hat{q}_1^*(\tau) - \hat{q}_0^*(\tau)$. The following theorem gives the weak convergence of the process $\{\sqrt{n}(\hat{q}^{\Delta}(\tau) - q^{\Delta}(\tau)) : \tau \in [\tau_L, \tau_U]\}$. Let $\ell^{\infty}([\tau_L, \tau_U])$ be the collection of bounded and measurable functions on $[\tau_L, \tau_U]$. Let BL_1 be the bounded Lipschitz functionals on $\ell^{\infty}([\tau_L, \tau_U])$ with Lipschitz constant 1, i.e.,

$$\mathrm{BL}_1 = \{h \in \ell^{\infty}([\tau_L, \tau_U]) : |h(\tau_1) - h(\tau_2)| \leqslant |\tau_1 - \tau_2|, \tau_1, \tau_2 \in [\tau_L, \tau_U]\}. \tag{78}$$

For a sequence of stochastic processes $\mathbb{G}_n$ and a process $\mathbb{G}$ on $[\tau_L, \tau_U]$, we write

$$\mathbb{G}_n \rightsquigarrow \mathbb{G}, \text{ in } \ell^{\infty}[\tau_L, \tau_U]), \tag{79}$$

as $n \to \infty$, if

$$\sup_{h \in \mathrm{BL}_1} |E^*[h(\mathbb{G}_n)] - E[h(\mathbb{G})]| \to 0, \tag{80}$$

as $n \to \infty$, where E^* denotes the outer expectation. Let $\mathbb{G}_n^*$ be a stochastic process on $[\tau_L, \tau_U]$ such that for each $\tau \in [\tau_L, \tau_U]$, $\mathbb{G}_n^*(\tau)$ is a measurable map of the bootstrap sample (Y_i^*, X_i^*). Then if for any $\varepsilon > 0$,

$$P^*\left\{\sup_{h \in \mathrm{BL}_1} |E[h(\mathbb{G}_n^*)|\mathcal{F}_n] - E[h(\mathbb{G})]| > \varepsilon\right\} \to 0, \tag{81}$$

as $n \to \infty$, for some we write $\mathbb{G}_n^* \rightsquigarrow_* \mathbb{G}$ in $\ell^{\infty}([\tau_L, \tau_U])$. Here P^* denotes the outer probability.

Theorem B.2 *Suppose that Assumptions 2.2 and 2.3 in the main text hold. Then the following statements hold.*

(i)

$$\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}) \rightsquigarrow \mathbb{G}, \text{ in } \ell^{\infty}([\tau_L, \tau_U]). \tag{82}$$

(ii)

$$\sqrt{n}(\hat{q}^{\Delta *} - \hat{q}^{\Delta}) \rightsquigarrow_* \mathbb{G}, \ \ in \ \ell^\infty([\tau_L, \tau_U]). \tag{83}$$

Proof: Define

$$\mathcal{G} = \{\xi(\cdot; q, \tau) : (q, \tau) \in J_d(\tau_U, \tau_L) \times [\tau_L, \tau_U]\}, \tag{84}$$

where

$$\xi(V_j; q, \tau) = -\frac{a_\tau(Y_j; q)1\{D_i = d\}}{f_d(q)p_d(X_i)} \tag{85}$$

$$+E\left[\frac{a_\tau(Y_i; q)g_d(X_i; \beta_0)'1\{D_i = d\}}{f_d(q)p_d^2(X_i)}\right]\zeta_j + o_P(1). \tag{86}$$

For (i), it suffices to show that $\mathcal{G}$ is P-Donsker. This also implies (ii) by Theorem 2.2 of Giné (1997, p. 104). Define

$$\nu_n(\xi) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\xi(V_i) - E\xi(V_i)). \tag{87}$$

The convergence of finite dimensional distributions of $\{\nu_n(\xi) : \xi \in \mathcal{G}\}$ follow by the usual central limit theorem. In order to show that $\mathcal{G}$ is P-Donsker, it suffices to show that $\mathcal{G}$ is totally bounded with respect to a pseudo-norm ρ and $\{\nu_n(\xi) : \xi \in \mathcal{G}\}$ is asymptotically equicontinuous with respect to ρ. We take the norm ρ to be $\|\cdot\|_{P,2}$. The total boundedness follows by the same arguments as in the proof of (i) of Lemma B.2. It remains to show asymptotic equicontinuity of the process ν_n. For this, we write

$$\sqrt{n}(\hat{q}^{\Delta}(\tau) - q^{\Delta}(\tau)) = -A_{n,1}(q_d(\tau), \tau) + A_{n,2}(q_d(\tau), \tau) + o_P(1), \tag{88}$$

where

$$A_{n,1}(q, \tau) = \frac{1}{\sqrt{n}f_d(q)}\sum_{j=1}^{n}\frac{a_\tau(Y_j; q)1\{D_j = d\}}{p_d(X_j)}, \text{ and} \tag{89}$$

$$A_{n,2}(q, \tau) = \frac{1}{\sqrt{n}f_d(q)}\sum_{j=1}^{n}E\left[\frac{a_\tau(Y_i; q)g_d(X_i; \beta_0)'1\{D_i = d\}}{p_d^2(X_i)}\right]\zeta_j. \tag{90}$$

Stochastic equicontinuity of $\{A_{n,1}(q,\tau) : (q,\tau) \in J_d(\tau_U,\tau_L) \times [\tau_L,\tau_U]\}$ obviously follows from Lemma B.2(i) and the Lipshitz continuity of $1/f_d(q)$ in $q \in J_d(\tau_U,\tau_L)$. It is not hard to show similarly that $A_{n,2}$ is stochastically equicontinuous as well. ■

We are prepared to prove Theorem 2.1 in the main text.

Proof of Theorem 2.1 in the main text:[3] For the proof, we follow arguments similar to Romano and Shaikh (2010). Define for any $W' \subset W$,

$$\hat{c}_{1-\alpha}(W') = \inf\left\{c \in \mathbb{R} : \frac{1}{B}\sum_{b=1}^{B} \mathbf{1}\left\{T_b^*(W') \leqslant c\right\} \geqslant 1-\alpha\right\}.$$

In light of Theorem 2.1 of Romano and Shaikh (2010) and the fact that the functional $\sup_{w \in W'} \Gamma(\cdot, S_w)$ is increasing in W', it suffices to show that

$$\limsup_{n\to\infty} P\left\{\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) > \hat{c}_{1-\alpha}(W_P)\right\} \leqslant \alpha. \tag{91}$$

By the continuous mapping theorem, we have

$$\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) \to_d \sup_{w \in W_P} \Gamma(\mathbb{G}; S_w),$$

as $n \to \infty$. Similarly, by the continuous mapping theorem of the bootstrap empirical process (e.g. Varron (2019)), we obtain that

$$\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta *} - q^{\Delta *}); S_w) \to_d^* \sup_{w \in W_P} \Gamma(\mathbb{G}; S_w),$$

as $n \to \infty$, where $\to_d^*$ denotes the convergence in bootstrap distribution in probability. Since $\sup_{w \in W_P} \Gamma(\cdot; S_w)$ is a nonnegative convex functional, its CDF is continuous and strictly increasing on $(0,\infty)$ by Theorem 11.1 of Davydov, Lifshits, and Smorodina (1998). Let $c_{1-\alpha}(W_P)$ be the $1-\alpha$ quantile of the distribution of $\sup_{w \in W_P} \Gamma(\mathbb{G}; S_w)$. By Theorem 1.2.1 of Politis, Romano, and Wolf (1999),

$$\hat{c}_{1-\alpha}(W_P) \to_P c_{1-\alpha}(W_P),$$

[3]We thank an anonymous referee for pointing out a gap in the proof in our previous manuscript.

as $n \to \infty$. Hence for any $\epsilon > 0$,

$$\begin{aligned} & P\left\{\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) > c_{1-\alpha}(W_P) + \epsilon\right\} + o(1) \\ & \leqslant P\left\{\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) > \hat{c}_{1-\alpha}(W_P)\right\} \\ & \leqslant P\left\{\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) > c_{1-\alpha}(W_P) - \epsilon\right\} + o(1). \end{aligned} \tag{92}$$

However,

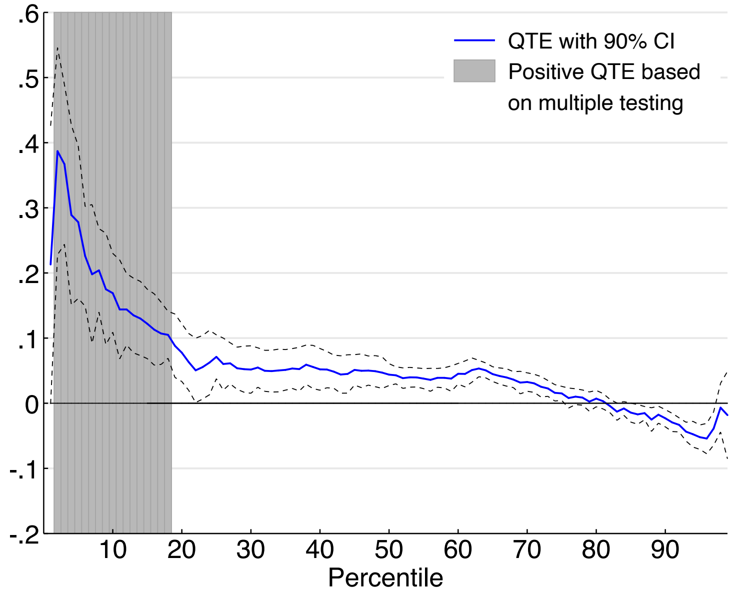
$$\begin{aligned} & P\left\{\sup_{w \in W_P} \Gamma(\sqrt{n}(\hat{q}^{\Delta} - q^{\Delta}); S_w) > c_{1-\alpha}(W_P) + \epsilon\right\} \\ & = P\left\{\sup_{w \in W_P} \Gamma(\mathbb{G}; S_w) > c_{1-\alpha}(W_P) + \epsilon\right\} \to \alpha, \end{aligned}$$

as $n \to \infty$ and then $\epsilon \to 0$. Since the CDF of $\sup_{w \in W_P} \Gamma(\mathbb{G}; S_w)$ is continuous, we apply the same argument to the upper bound in (92), we obtain (91). ■

C Robustness checks

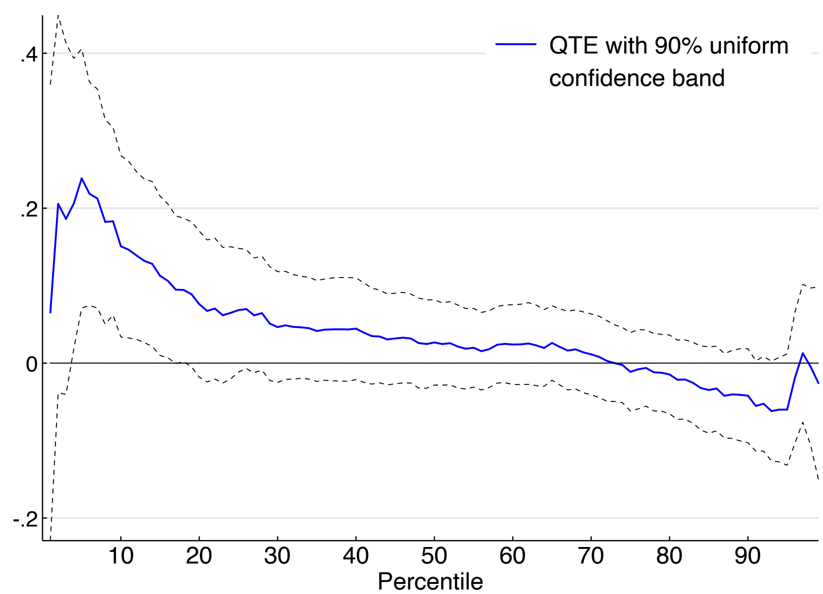
In this section, we present two additional analyses as robustness checks to our main multiple testing results without subgroups (see Figure 1 in the paper). First, to ensure that type I error rates and power are uniform across quantiles, we use the bootstrap interquartile range rescaling in Algorithm 3 of Chernozhukov, Fernandez-Val, and Melly (2013) before conducting the step-down procedure. The multiple testing results in Figure 2 reveal that, in contrast to our main results in Figure 1 of the paper, we cannot reject the null hypothesis at the 1st and 19th percentiles. Hence, overall, the bootstrap interquartile range rescaling does not change our findings.

Second, we implement the procedure proposed by Chernozhukov, Fernandez-Val, and Melly (2013), i.e. we estimate conditional QTE and construct uniform confidence bands, as an alternative way to determine for which quantiles the null hypothesis of non-positive QTE is violated. Figure 3 presents the results. While the shape of the estimated QTE is not identical to Figure 1 in the paper because the Chernozhukov, Fernandez-Val, and Melly (2013) procedure is based on conditional QTE, we observe a similar general pattern of treatment effects that decline in magnitude at higher quantiles that also become statistically insignificant. We find that the null hypothesis is violated for the 5th to the 17th and the



Note: Multiple testing results show quantiles for which the QTE is positive at an FWER of 5 percent (see hypothesis (H.3) in Section 2.3 of the main text).

Figure 2: QTE and multiple testing results using the bootstrap interquartile range rescaling, no subgroups



Note: QTEs and confidence set estimated following Chernozhukov, Fernandez-Val, and Melly (2013).

Figure 3: QTE and uniform confidence bands, no subgroups

19th percentiles, which aligns well with our main result. In summary, these robustness checks reinforce our main findings that ignoring multiple testings leads to substantial inflation in the number of false positive conclusions and that the set of significantly positive QTE at lower percentiles supports the distributional effects predicted by the underlying theory.

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